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The Weierstrass-Stone Theorem in absolute valued division rings

by João B. Prolla

Departamento de Matemática, IMECC-UNICAMP, Caixa Postal 6065, 13081 Campinas, SP, Brazil

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ABSTRACT

Let S be a zero-dimensional compact Hausdorff space and let E be a normed space over a non-Archimedean absolute valued division ring $(\mathbb{K}, |\cdot|)$. The space $C(S;E)$ of all continuous functions from S into E is equipped with the uniform topology given by the supremum norm. A Weierstrass-Stone Theorem for arbitrary subsets of $C(S;E)$ is established.

1. THE MAIN THEOREM

Let S be a compact Hausdorff space which is 0-dimensional (i.e., for any point s belonging to an open subset G, there exists a closed and open set A with $s \in A \subset G$). Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean absolute valued division ring. Let E be a non-trivial normed space over $\mathbb{K}$, and let $C(S;E)$ be the linear space of all continuous functions from S into E, equipped with the supremum norm

$$\|f\| = \sup\{\|f(x)\|;\ x \in S\}.$$

DEFINITION 1. A non-empty subset $M \subset C(S;\mathbb{K})$ is said to have *property V* if

(1) $|\varphi(s)| \leq 1$, for every $s \in S$ and $\varphi \in M$;

(2) if $\varphi \in M$, then $1-\varphi$ belongs to M;

(3) if φ and ψ belong to M, then $\varphi\psi \in M$.

DEFINITION 2. Let $W \subset C(S;E)$ be a non-empty subset. A function $\varphi \in C(S;\mathbb{K})$ is called a *multiplier* of W if

(1) $|\varphi(s)| \leq 1$, for every $s \in S$;

(2) if f and g belong to W, then $\varphi f + (1-\varphi)g$ belongs to W.

Clearly, if M denotes the set of all multipliers of W, then M satisfies conditions (1) and (2) of Definition 1. The identity

$$(\varphi\psi)f + (1-\varphi\psi)g = \varphi[\psi f + (1-\psi)g] + (1-\varphi)g$$

shows that M satisfies condition (3) as well. Hence M has property V. Notice that the constant functions 0 and 1 belong to M.

DEFINITION 3. A subset $A \subset C(S; \mathbb{K})$ is said to be *separating over* S, if given any two distinct points, s and t, of S, there exists a function $\varphi \in A$ such that $\varphi(s) \neq \varphi(t)$.

DEFINITION 4. A subset $M \subset C(S; \mathbb{K})$ is said to be *strongly separating over* S, if given any ordered pair $(s, t) \in S \times S$, with $s \neq t$, there exists a function $\varphi \in M$ such that $\varphi(s) = 1$, $\varphi(t) = 0$ and $|\varphi(x)| \leq 1$ for all $x \in S$.

PROPOSITION 1. *If A is a unitary subalgebra of $C(S; \mathbb{K})$ which is separating over S, then A is strongly separating over S.*

PROOF. Let $s \neq t$ be given in S. Since A is vector space containing the constants, there is $a \in A$ such that $a(s) = 1$ and $a(t) = 0$. Since a is continuous, $a(S)$ is a compact subset of $\mathbb{K}$. By Kaplansky's Lemma (see Lemma 1.23, Prolla [7]) there is a polynomial $p: \mathbb{K} \to \mathbb{K}$ such that $p(1) = 1$, $p(0) = 0$ and $|p(y)| \leq 1$ for all $y \in a(S)$. Let $\varphi(x) = p(a(x))$, for all $x \in S$. Then $\varphi \in A$, $\varphi(s) = 1$, $\varphi(t) = 0$, and $|\varphi(x)| \leq 1$ for all $x \in S$. Hence A strongly separates the points of S. □

LEMMA 1. *Let $M \subset C(S; \mathbb{K})$ be a non-empty subset with property V, and containing the constant functions* 0 *and* 1. *Assume that M is strongly separating over S. Let N be a clopen subset of S. For each $\delta > 0$, there is $\varphi \in M$ such that*

(1) $|1 - \varphi(t)| < \delta$, *for all* $t \in N$,

(2) $|\varphi(t)| < \delta$, *for all* $t \notin N$.

PROOF. If $N = S$, the function $\varphi(t) = 1$ for all $t \in S$, satisfies (1) and (2).

If $N = \emptyset$, the function $\varphi(t) = 0$ for all $t \in S$ satisfies (1) and (2). Assume that $K = S \setminus N$ is non-empty. Fix $y \in S$, $y \notin N$. For each $t \in N$, there is $\varphi_t \in M$ such that $\varphi_t(t) = 0$, $\varphi_t(y) = 1$. By continuity there exists a neighborhood $V(t)$ of t such that $|\varphi_t(s)| < \delta$ for all $s \in V(t)$. By compactness of N there are $t_1, \ldots, t_n \in N$ such that $N \subset V(t_1) \cup \cdots \cup V(t_n)$. Consider $\varphi_y = 1 - \varphi_{t_1} \cdot \varphi_{t_2} \cdot \cdots \cdot \varphi_{t_n}$. Then $\varphi_y \in M$ and $\varphi_y(y) = 0$, while $|1 - \varphi_y(t)| < \delta$, for all $t \in N$. Indeed, if $t \in N$, then $t \in V(t_i)$ for some $i = 1, \ldots, n$. Hence

$$|1 - \varphi_y(t)| = |\varphi_{t_i}(t)| \cdot \prod_{j \neq i} |\varphi_{t_j}(t)| < \delta.$$

By continuity there exists a neighborhood $W(y)$ of y such that $|\varphi_y(s)| < \delta$ for all $s \in W(y)$. By compactness of K, there are $y_1, \dots, y_m \in K$ such that $K \subset W(y_1) \cup \cdots \cup W(y_m)$. Let $\varphi = \varphi_{y_1} \cdot \varphi_{y_2} \cdot \cdots \cdot \varphi_{y_m}$. Clearly, $\varphi \in M$. We claim that for each $k = 1, 2, \dots, m$, we have

(3) $\qquad |1 - \varphi_{y_1}(t)\varphi_{y_2}(t) \cdot \cdots \cdot \varphi_{y_k}(t)| < \delta, \quad$ for all $t \in N$.

Clearly, (1) follows from (3) by taking $k = m$. We prove (3) by induction. For $k = 1$, (3) is clear, since $|1 - \varphi_y(t)| < \delta$ for all $t \in N$ and $y \in K$. Assume (3) has been proved for k. To simplify notation we write $\varphi_i = \varphi_{y_i}$ for all $1 \le i \le m$. Then, for each $t \in N$,

$$\begin{aligned} &|1 - \varphi_1(t) \cdot \cdots \cdot \varphi_{k+1}(t)| \\ &\quad = |1 - \varphi_{k+1}(t) + \varphi_{k+1}(t) - \varphi_1(t) \cdot \cdots \cdot \varphi_k(t) \cdot \varphi_{k+1}(t)| \\ &\quad \le \max(|1 - \varphi_{k+1}(t)|, |\varphi_{k+1}(t)| \cdot |1 - \varphi_1(t) \cdot \cdots \cdot \varphi_k(t)|) < \delta, \end{aligned}$$

because $|1 - \varphi_{k+1}(t)| < \delta$, $|\varphi_{k+1}(t)| \le 1$, and by the induction hypothesis, $|1 - \varphi_1(t) \cdot \cdots \cdot \varphi_k(t)|) < \delta$. Hence (3) is true for $k + 1$.

It remains to prove (2), i.e. $|\varphi(t)| < \delta$ for all $t \in K$. Now, if $t \in K$, then $t \in W(y_i)$ for some $i = 1, \dots, m$. Hence $|\varphi_i(t)| < \delta$, while $|\varphi_j(t)| \le 1$ for all $j \ne i$. Therefore $|\varphi(t)| < \delta$, and (2) is proved. □

THEOREM 1. *Let W be a non-empty subset of $C(S; E)$ such that the set M of all multipliers of W strongly separates the points of S. Let $f \in C(S; E)$ and $\varepsilon > 0$ be given. The following are equivalent*:

(1) *there is some $g \in W$ such that $\|f - g\| < \varepsilon$;*

(2) *for each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$.*

PROOF. Clearly (1) $\Rightarrow$ (2). Conversely, assume that (2) is true. For each $x \in S$, there is some $g_x \in W$ such that $\|f(x) - g_x(x)\| < \varepsilon$. Choose a real number $\varepsilon(x) > 0$ such that $\|f(x) - g_x(x)\| < \varepsilon(x) < \varepsilon$. Let $N(x)$ be a clopen neighborhood of x in S such that

$$N(x) \subset \{t \in S; \|f(t) - g_x(t)\| < \varepsilon(x)\}.$$

Select a point $x_1 \in S$ arbitrarily. Let $K = S \setminus N(x_1)$. By compactness of K, there exists a finite set $\{x_2, \dots, x_m\} \subset K$ such that $K \subset N(x_2) \cup \cdots \cup N(x_m)$. Let

$$\begin{aligned} &N_2 = N(x_2) \setminus N(x_1), \\ &N_3 = N(x_3) \setminus (N(x_1) \cup N(x_2)), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &N_m = N(x_m) \setminus (\bigcup_{j=1}^{m-1} N(x_j)). \end{aligned}$$

Then $N_2, N_3, \dots, N_m$ are clopen subsets of S, such that $K \subset N_2 \cup N_3 \cup \cdots \cup N_m$, and $N_i \cap N_j = \emptyset$ for all $i \ne j$ $(2 \le i, j \le m)$. Let us write $g_i = g_{x_i}$ for all $i = 1, 2, \dots, m$, and let

$$k = \max\{\|f - g_1\|, \|f - g_2\|, \dots, \|f - g_m\|\}.$$

Choose a number $\delta > 0$ so small that $\delta k(m-1) < \varepsilon - \varepsilon'$, where $\varepsilon' = \max\{\varepsilon(x_1), \varepsilon(x_2), \ldots, \varepsilon(x_m)\}$. By Lemma 1, there are $\varphi_2, \ldots, \varphi_m \in M$ such that

(1) $\quad |1 - \varphi_i(t)| < \delta, \quad$ for all $t \in N_i$

(2) $\quad |\varphi_i(t)| < \delta, \quad$ for all $t \notin N_i$

for all $i = 2, \ldots, m$. Define $N_1 = N(x_1)$, and

$$\begin{aligned} \psi_2 &= \varphi_2 \\ \psi_3 &= (1-\varphi_2)\varphi_3 \\ &\cdots\cdots\cdots\cdots \\ \psi_m &= (1-\varphi_2)(1-\varphi_3)\cdot\cdots\cdot(1-\varphi_{m-1})\varphi_m. \end{aligned}$$

Clearly, $\psi_i \in M$, for all $i = 2, 3, \ldots, m$. Now

$$\psi_2 + \cdots + \psi_j = 1 - (1-\varphi_2)(1-\varphi_3)\cdot\cdots\cdot(1-\varphi_j), \quad j = 2, \ldots, m,$$

can be easily verified by induction. Define

$$\psi_1 = (1-\varphi_2)(1-\varphi_3)\cdots(1-\varphi_m).$$

Then $\psi_1 \in M$ and $\psi_1 + \psi_2 + \cdots + \psi_m = 1$. Notice that

(3) $\quad |\psi_i(t)| < \delta, \quad$ for all $t \notin N_i, \quad i = 1, 2, \ldots, m$.

Indeed, if $i \geq 2$, then $|\psi_i(t)| \leq |\varphi_i(t)|$ and (3) follows from (2). If $i = 1$, and $t \notin N(x_1)$, then $t \in K$. Hence $t \in N_j$ for some $j = 2, \ldots, m$. By (1), $|1 - \varphi_j(t)| < \delta$ and so

$$|\psi_1(t)| = |1 - \varphi_j(t)| \cdot \prod_{j \neq i} |1 - \varphi_i(t)| < \delta,$$

because $|1 - \varphi_i(t)| \leq 1$ for all $i \neq j$. Let $g = \psi_1 g_1 + \psi_2 g_2 + \cdots + \psi_m g_m$. Then

$$\begin{aligned} g = \varphi_2 g_2 &+ (1-\varphi_2) \\ &\cdot [\varphi_3 g_3 + (1-\varphi_3)[\varphi_4 g_4 + \cdots + (1-\varphi_{m-1})[\varphi_m g_m + (1-\varphi_m)g_1]\cdots]]. \end{aligned}$$

Hence $g \in W$. Let $x \in S$ be given. There is exactly one integer $1 \leq i \leq m$ such that $x \in N_i$. Call it j. Then $|\psi_j(x)| \cdot \|f(x) - g_j(x)\| \leq |\psi_j(x)| \cdot \varepsilon(x_j) < \varepsilon'$, since $|\psi_j(x)| \leq 1$. For all $i \neq j$, we have that $x \notin N_i$. By (3), $|\psi_i(x)| < \delta$. Hence

$$\sum_{i \neq j} |\psi_i(x)| \cdot \|f(x) - g_i(x)\| \leq \delta k(m-1) < \varepsilon - \varepsilon',$$

and therefore

$$\begin{aligned} \|f(x) - g(x)\| &= \left\| \sum_{i=1}^{m} \psi_i(x)(f(x) - g_i(x)) \right\| \\ &\leq \varepsilon' + \sum_{i \neq j} |\psi_i(x)| \cdot \|f(x) - g_i(x)\| < \varepsilon' + \varepsilon - \varepsilon' = \varepsilon. \qquad \square \end{aligned}$$

REMARK. To give examples of non-commutative division rings let us consider the so-called *quaternion algebras*. Let p be an odd prime, and let Q_p be the p-adic field, i.e., the completion of the rational field Q with the p-adic absolute value. Let a and b be non-zero elements of Q_p. Let $A(a,b,p)$ be the four dimensional vector space over Q_p with basis $\{1,i,j,k\}$ and the bilinear multiplication defined by the conditions that 1 is a unity and $i^2=a$, $j^2=b$, $k^2=-ab$, $ij=-ji=k$, $ik=-ki=ja$, $jk=-kj=-ib$. With this structure, $A(a,b,p)$ is an associative Q_p-algebra. The following conditions are equivalent (see Pierce [6], p. 15):

(1) *$A(a,b,p)$ is a division algebra*
(2) *for every $x_0,x_1,x_2\in Q_p$, if $x_0^2=ax_1^2+bx_2^2$ then $x_0,x_1,x_2=0$.*

For example, if $a\in\mathbb{N}$ is not a quadratic residue modulo p, and $b=p$, then the quaternion algebra $D=A(a,p,p)$ is a division algebra. On the other hand, when $p=5$, the existence of $\sqrt{-1}\in Q_5$ shows that $A(-1,-1,5)$ is not a division algebra, since $x_0=0$, $x_1=1$, $x_2=\sqrt{-1}$ satisfy $x_0^2+x_1^2+x_2^2=0$.

In fact, by the results of chapter IV, §2, of Serre [9], up to isomorphisms, there is a unique non-commutative division algebra of dimension four over Q_p, given by the quaternion algebra $A(a,b,p)$, when the pair (a,b) satisfies condition (2) above. By Theorem 26.5 of Warner [10], there is a non-archimedean absolute value on the division ring $\mathbb{K}=A(a,b,p)$ which extends the p-adic absolute value of Q_p. Just notice that Q_p is complete and $\mathbb{K}=A(a,b,p)$ is a finite dimensional division algebra over Q_p. (Of course we are assuming that a and b satisfy condition (2).) For each $z\in\mathbb{K}$, let $N(z)$ be the determinant of the linear operator L_z defined by $L_z(x)=zx$, for all $x\in\mathbb{K}$. Notice that $N(z)\in Q_p$ and then

$$|z|_p:=|N(z)|_p^{1/4}$$

is the non-Archimedean absolute value on $\mathbb{K}=A(a,b,p)$ extending the p-adic absolute value $|\cdot|_p$ of Q_p.

2. SOME CONSEQUENCES

Let us recall the definition of the distance of an element $f\in C(S;E)$ from W:

$$\operatorname{dist}(f;W)=\inf\{\|f-g\|;\ g\in W\}.$$

THEOREM 2. *Let W be a non-empty subset of $C(S;E)$ such that the set M of all multipliers of W strongly separates the points of S. For each $f\in C(S;E)$ there exists $x\in S$ such that*

$$\operatorname{dist}(f;W)=\operatorname{dist}(f(x);W(x)).$$

PROOF. If $\operatorname{dist}(f;W)=0$, then $\operatorname{dist}(f(x);W(x))=0$ for every $x\in S$. Suppose now that $\operatorname{dist}(f;W)=d>0$. By contradiction, assume that $\operatorname{dist}(f(x);W(x))<d$ for every $x\in S$. Hence, for each $x\in S$, there is some $g_x\in W$ such that

$\|f(x)-g_x(x)\|<d$. Consequently, f and $d>0$ satisfy condition (2) of Theorem 1. By Theorem 1, there exists $g\in W$ such that $\|f-g\|<d$, a contradiction, since $d=\operatorname{dist}(f;W)$. □

THEOREM 3 (Kaplansky [4]). *Let A be a unitary subalgebra of $C(S;\mathbb{K})$ which is separating over S. Then A is uniformly dense in $C(S;\mathbb{K})$.*

PROOF. Let $E=\mathbb{K}$ and $W=A$. Notice that every element $\varphi\in A$, such that $|\varphi(x)|\leq 1$ for all $x\in S$, is a multiplier of W. By Proposition 1, the set M of all multipliers of W is strongly separating over S. Let now $f\in C(S;\mathbb{K})$ be given. By Theorem 2, there exists $x\in S$ such that

$$\operatorname{dist}(f;A)=\operatorname{dist}(f(x);A(x)).$$

Since A contains the constants, $A(x)=\mathbb{K}$. Hence $\operatorname{dist}(f(x);A(x))=0$, and therefore $\operatorname{dist}(f;A)=0$. This shows that A is uniformly dense in $C(S;\mathbb{K})$. □

COROLLARY 1 (Weierstrass Theorem). *Let S be a non-empty compact subset of $\mathbb{K}$. For every $f\in C(S;\mathbb{K})$ and every $\varepsilon>0$, there exists a polynomial p with coefficients in $\mathbb{K}$ such that $|f(x)-p(x)|<\varepsilon$, for all $x\in S$.*

REMARK. When $\mathbb{K}$ is the field of p-adic numbers with the p-adic valuation, Theorem 3 and its Corollary 1 were proved by J. Dieudonné in 1944. (See Dieudonné [2].) In 1947, I. Kaplansky showed that a Weierstrass-Stone theorem holds for functions with values in topological rings having ideal neighborhoods of 0. (See Kaplansky [3].) Now in $\mathbb{K}$, the set $\{\lambda\in\mathbb{K};|\lambda|<1\}$, called the valuation ideal of $\mathbb{K}$, is an ideal neighborhood of 0. In 1950, Kaplansky showed that the methods of [3] could be extended to $(\mathbb{K},|\cdot|)$ by proving Theorem 3. In fact, he proved a more general version of Theorem 3, by considering S to be a 0-dimensional locally compact Hausdorff space, and $C_0(S;\mathbb{K})$ the space of all those $f\in C(S;\mathbb{K})$ vanishing at infinity, and $A\subset C_0(S;\mathbb{K})$ a subalgebra containing for any two distinct points $s,t\in S$ a function vanishing at s but not at t. (See Kaplansky [4]). In 1958, K. Mahler gave a constructive proof of Dieudonné's Weierstrass theorem (Corollary 1 above) for the case S is the ring of p-adic integers $\{\lambda\in Q_p;|\lambda|_p\leq 1\}$. (See Mahler [5].) However, Mahler's proof is based on some properties of the cyclotomic extension of Q. In 1974, R. Bojanic presented another proof of Mahler's result, which is entirely analytic. (See Bojanic [1].)

3. SIMULTANEOUS APPROXIMATION AND INTERPOLATION

DEFINITION 5. A non-empty subset $A\subset C(S;E)$ is called an *interpolating family for* $C(S;E)$ if, for every $f\in C(S;E)$ and every finite subset $F\subset S$, there exists $g\in A$ such that $f(x)=g(x)$ for all $x\in F$.

Let us study the problem of simultaneous approximation and interpolation. We start with scalar-valued functions, i.e., subsets of $C(S;\mathbb{K})$.

THEOREM 4. *Let A be a uniformly dense linear subspace of $C(S;\mathbb{K})$. Then, for every $f \in C(S;\mathbb{K})$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f-g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

PROOF. Let $F = \{x_1, \ldots, x_n\}$. Define a linear mapping $T: C(S;\mathbb{K}) \to \mathbb{K}^n$ by

$$Tg = (g(x_1), \ldots, g(x_n))$$

for each $g \in C(S;\mathbb{K})$. By density of A and continuity of T, we have

$$T(C(S;\mathbb{K})) = T(\bar{A}) \subset \overline{T(A)}.$$

Now $T(A)$ is a linear subspace of $\mathbb{K}^n$ and therefore $T(A)$ is closed. Hence

$$T(C(S;\mathbb{K})) = T(A)$$

and A is an interpolating family for $C(S;\mathbb{K})$. Therefore $a_1, \ldots, a_n$ can found in A such that

$$a_i(x_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

Choose $\delta > 0$ so that $\delta < \varepsilon$ and $\delta k < \varepsilon$, where $k = \max\{\|a_i\|; 1 \le i \le n\}$. By density of A there is some $g_1 \in A$ such that $\|f - g_1\| < \delta$. Let

$$v_i = f(x_i) - g_1(x_i), \quad 1 \le i \le n.$$

Define $g_2 = \sum_{i=1}^{n} v_i a_i$. Then $g_2 \in A$ and $g_2(x_j) = v_j$ for all $1 \le j \le n$. Finally, let $g = g_1 + g_2$. Then $g \in A$ and $g(x_j) = f(x_j)$, $1 \le j \le n$. Moreover,

$$\|f - g\| \le \max(\|f - g_1\|, \|g_2\|) < \varepsilon,$$

since $\|f - g_1\| < \varepsilon$ and $\|g_2\| \le \delta \max\{\|a_i\|; 1 \le i \le n\}$. □

COROLLARY 2. *Let A be a unitary subalgebra of $C(S;\mathbb{K})$ which is separating over S. Then, for every $f \in C(S;\mathbb{K})$, every $\varepsilon > 0$ and every finite subset $F \subset S$, there exists $g \in A$ such that $\|f-g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

PROOF. By Theorem 3, A is a uniformly dense linear subspace of $C(S;\mathbb{K})$. It remains to apply Theorem 4. □

REMARK. The proof of Theorem 4 does not extend to subsets of $C(S;E)$. In this case we rely on Theorem 1, as our next result shows.

THEOREM 5. *Let $A \subset C(S;E)$ be an interpolating family for $C(S;E)$ such that the set of multipliers of A strongly separates the points of S. Then, for every $f \in C(S;E)$, every $\varepsilon > 0$ and every finite subset $F \subset S$; there exists $g \in A$ such that $\|f-g\| < \varepsilon$ and $f(x) = g(x)$ for all $x \in F$.*

PROOF. Let $W = \{g \in A; f(x) = g(x) \text{ for all } x \in F\}$. Since A is an interpolating family, $W \ne \emptyset$. Notice that every multiplier of A is also a multiplier of W. Let $x \in S$ be given. Consider the finite set $F \cup \{x\}$. Since A is an interpolating family for $C(S;E)$, there exists $g_x \in A$ such that $f(t)=g_x(t)$ for all $t \in F \cup \{x\}$. Therefore $g_x \in W$. Notice that $\|f(x)-g_x(x)\|=0<\varepsilon$. By Theorem 1 there exists $g \in W$ such that $\|f-g\|<\varepsilon$. Notice that $g \in W$ implies $g \in A$ and $f(x)=g(x)$ for all $x \in F$. $\square$

In a forthcoming paper [8] we show how to extend some of the results of this paper to the case of a topological ring (E, τ).

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